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L^p-Spaces for C^*-Algebras with a State†

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We present here a construction of noncommutative L^p-spaces for a C^*-algebra with respect to a state on the algebra. Their properties are deduced from well-established properties of corresponding Haagerup and Kosaki spaces. Two examples are considered.

1. INTRODUCTION

There have been several attempts at the construction of noncommutative L^p-spaces for specific C^*-algebras (Majewski and Zegarliński, 1995, 1996; Goldstein and Phan, 1998). We give here a general definition and the basic properties of the spaces. Further details may be found in Phan (1999).

Let M be a von Neumann algebra acting in a Hilbert space H and ψ a normal faithful semifinite weight on M. Let $\{\sigma_t^\psi\}_{t\in\mathbb{R}}$ denote the modular automorphism group on M associated with ψ. Recall that the crossed product $\mathbb{M} = M \rtimes_{\sigma_\psi} \mathbb{R}$ is a von Neumann algebra acting on $\mathbb{H} = L^2(\mathbb{R}, H)$ generated by operators $\pi_M(a)$, $a \in M$ and $\lambda_M(s)$, $s \in \mathbb{R}$, defined by

$$(\pi_M(a)\xi)(t) = \sigma_{-t}^\psi(a)\xi(t) \quad (\lambda_M(s)\xi)(t) = \xi(t-s) \quad \xi \in \mathbb{H},\ t \in \mathbb{R}.$$

Let τ denote the canonical trace on $\mathbb{M}$. Denote by $\tilde{\mathbb{M}}$ the algebra of all τ-measurable operators affliated with $\mathbb{M}$. The dual representation (or the *dual action* of $\mathbb{R}$ on $M \rtimes_\sigma \mathbb{R}$) is the continuous automorphism representation $s \mapsto \theta_s$ of $\mathbb{R}$ where θ_s is the *automorphism of $M \rtimes_\sigma \mathbb{R}$ that is implemented by the unitary operator μ_s defined by $(\mu_s\xi)(t) = e^{-its}\xi(t)$, $\xi \in L^2(\mathbb{R}, H)$, $t \in \mathbb{R}$. For each $p \in [1, \infty]$, the Haagerup L^p-space is defined by

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$$L^p(M) := \{h \in \tilde{\mathbb{M}}: \forall s \in \mathbb{R}, \theta_s h = e^{-s/p} h\}.$$

We identify $L^\infty(M)$ with M by means of π_M.

Lemma 1.1. Let M be a von Neumann algebra, φ a faithful normal state on M, σ_t^φ the modular automorphism group of M associated with φ, $\mathbb{M} = \mathrm{M} \rtimes_\sigma \mathbb{R}$, $\overline{\varphi}$ the dual weight, τ the canonical normal faithful semifinite trace on $\mathbb{M}$, $h_\varphi = d\overline{\varphi}/d\tau$. Then, for $p \in [1, \infty]$,

1. the mapping $i_p\ M \to L^p(M)$ defined by $a \mapsto i_p(a) = h_\varphi^{1/2p} \cdot a \cdot h_\varphi^{1/2p}$ is linear and injective;
2. if M_0 is σ-weakly dense in M, then $i_p(M_0) = h_\varphi^{1/2p} \cdot M_0 \cdot h_\varphi^{1/2p}$ is norm-dense in $L^p(M)$ for $p < \infty$ (σ-weakly dense for $p = \infty$).
3. We have

$$1 \leq r \leq p \leq \infty \Rightarrow \|i_r(x)\|_r \leq \|i_p(x)\|_p, \qquad x \in M.$$

The first part can be found, for example, in Goldstein and Lindsay, 1995. As for the second part of the lemma, it is easy to prove for $p = 1$ using duality. The proof for any $p \in [1, \infty]$ can be deduced from the Kaplansky density theorem and the inequality

$$\|h^{1/2p} \cdot a \cdot h^{1/2p}\|_p \leq \|h^{1/2} \cdot a \cdot h^{1/2}\|_1^{1/p} \|a\|_\infty^{1/q}$$

with $1/p + 1/q = 1$, to be found in Terp (1982) or Goldstein and Lindsay, 1999. The last part follows from Hölder's inequality since

$$i_r(x) = h^{1/2s}\, i_p(x) h^{1/2s}$$

where $s \in [1, \infty]$ satisfies $1/p + 1/s = 1/r$.

2. DEFINITION OF $L^p(A, \varphi)$ SPACES.

Let A be a C^*-algebra and φ a state on A; let $(H_\varphi, \pi_\varphi, \xi_\varphi)$ denote the GNS representation of A associated with φ. In this section we introduce the spaces $L^p(A, \varphi)$. Their properties will be given in the next section. First let us specify some notations that we shall use in the sequel.

1. $\omega_{\xi\varphi}$ is the vector state on $\mathbb{B}(H_\varphi)$ given by

$$\omega_{\xi\varphi}(a) = (a\xi_\varphi, \xi_\varphi),\ a \in \mathbb{B}(H_\varphi)$$

2. s_φ is the support of the state $\omega_{\xi\varphi}|\pi_\varphi(A)''$ on the von Neumann algebra $\pi_\varphi(A)''$.
3. H denotes the Hilbert space $s_\varphi H_\varphi$ (with the inner product inherited from H_φ).
4. M is the von Neumann algebra $s_\varphi \pi_\varphi(A)'' s_\varphi$ acting on H.

5. ω denotes the faithful normal state $\omega_{\xi_\varphi}|M$.
6. σ_t^ω is the modular automorphism group of M relative to ω.
7. $\mathbb{M}$ is the crossed product $M \rtimes_{\sigma_t^\omega} \mathbb{R}$ acting on the Hilbert space $\mathbb{H} = L^2(\mathbb{R}, H)$; the image $\pi_M(M)$ of M in $\mathbb{M}$ will be denoted by M, too
8. θ_s denotes the dual action of $\mathbb{R}$ on $\mathbb{M}$ or its extension to $\tilde{\mathbb{M}}$ - the topological *algebra of τ-measurable operators affiliated with $\mathbb{M}$.
9. τ is the canonical normal faithful semifinite trace on $\mathbb{M}$.
10. $L^p(M)$ with $p \in [1, \infty]$ is the Haagerup space (consisting of measurable operators affiliated with $\mathbb{M}$) with norm $\|\cdot\|_p$.
11. h_φ is the measurable operator affiliated with $\mathbb{M}$ defined by $h_\varphi = d\overline{\omega}/d\tau$, where $\overline{\omega}$ is the dual weight of ω.

Consider the map $\gamma_\varphi\colon A \to M$ given by

$$a \mapsto s_\varphi \pi_\varphi(a) s_\varphi.$$

This is a positive linear contraction with σ-weakly dense range. Let N_φ be the kernel of γ_φ and let $\tilde{\gamma}_\varphi$ denote the induced map $A/N_\varphi \to M$. Then N_φ is a closed involutive subspace of A, and the quotient space A/N_φ is a Banach space in the quotient norm, with positive elements of the form $[a]$, for $a \in A_+$. The injective linear map $\gamma_p := i_p \circ \tilde{\gamma}_\varphi : A/N_\varphi \to L^p(M)$ is positivity preserving, and Lemma 1.1 implies that it has norm-dense range for $p < \infty$ and σ-weakly dense range for $p = \infty$.

Norms are defined on A/N_φ by

$$\|[a]\|_p = \|i_p(\gamma_\varphi(a))\|_{L^p(M)},$$

the resulting normed space is denoted $L_0^p(A, \varphi)$. Thus, for $p < \infty$, $(L^p(M), \gamma_p)$ is a completion of $L_0^p(A, \varphi)$ in which the dense isometric embedding γ_p respects positivity.

In order to obtain compatible spaces we consider a different family of completions. Let $(L^1(A, \varphi), \kappa)$ be any completion of $L_0^1(A, \varphi)$. By Lemma 2.1, the norms on A/N_φ satisfy

$$\|[a]\|_r \leq \|[a]\|_p \text{ for } 1 \leq r \leq p \leq \infty.$$

Therefore completions $(L^p(A, \varphi), \kappa_p)$ of $L_0^p(A, \varphi)$ may be found satisfying

$$\kappa(A/N_\varphi) \subset L^p(A, \varphi) \subset L^r(A, \varphi) \subset L^1(A, \varphi)$$

for $1 \leq r \leq p < \infty$.

The positive elements of $L^p(A, \varphi)$, $p < \infty$, are given by

$$L_+^p(A, \varphi) = \text{closure in } L^p(A, \varphi) \text{ of } \kappa_p((A/N_\varphi)_+)$$

We denote by Γ_p the unique isometric isomorphism from $L^p(A, \varphi)$ to $L^p(M)$ extending the maps $\tilde{\gamma}_\varphi$ and γ_p. It is clearly positivity preserving.

Note that the mappings $\Gamma_p^{-1} \circ i_p$, $p < \infty$, do not depend on p. Define $L^\infty(A, \varphi)$ to be the image of M under the mappings. When restricted to $L^\infty(A, \varphi)$, the mapping $i_p^{-1} \circ \Gamma_p$ is an isometric isomorphism from $L^\infty(A, \varphi)$ to $L^\infty(M)$ which extends $\tilde{\gamma}_\varphi$ and γ_∞. We denote it by Γ_∞.

It follows that the L^p-spaces over a C^*-algebra with respect to a state inherit all the standard properties of duality, reflexivity and uniform convexity, and the Hölder and Clarkson inequalities, from the Haagerup L^p-spaces. Note also that $L^\infty(A, \varphi)$ and $L^1(A, \varphi)$ form a compatible pair of Banach spaces. In the next section we shall fix the multiplicative structure of the spaces and state some of the properties that relate to the structure. We shall also show that these L^p-spaces are complex interpolation spaces, by relating them to Kosaki's L^p-spaces.

3. THE PROPERTIES OF $L^p(A, \varphi)$ SPACES

Let $p, q, r \in [1, \infty]$ be such that $1/p + 1/q = 1/r$. For $a \in L^p(A, \varphi)$, $b \in L^q(A, \varphi)$ define $a \,{}_p\!\cdot_q\, b \in L^r(A, \varphi)$ by

$$a \,{}_p\!\cdot_q\, b := \Gamma_r^{-1}\,(\Gamma_p(a) \cdot \Gamma_q(b)).$$

Proposition 3.1 (Hölder's inequality). Let $r, p, q \in [1, \infty]$ be such that $1/p + 1/q = 1/r$, $a \in L^p(A, \varphi)$, $b \in L^q(A, \varphi)$. Then

$$\|a \,{}_p\!\cdot_q\, b\|_r \leq \|a\|_p \|b\|_q.$$

We define a linear functional $\dot{\mathrm{tr}}$ on $L^1(A, \varphi)$ by

$$\dot{\mathrm{tr}}(a) = \mathrm{tr}(\gamma_1(a)), \; a \in L^1(A, \varphi),$$

where tr is the usual linear functional tr on $L^1(M)$. For $p, q \in [1, \infty]$, $1/p + 1/q = 1$, $a, b \in A/N_\varphi$ define

$$\langle a|b\rangle := \dot{\mathrm{tr}}(a \,{}_p\!\cdot_q\, b).$$

Proposition 3.2. Let $p, q \in [1, \infty]$, $1/p + 1/q = 1$. It follows that

1. $\dot{\mathrm{tr}}(a \,{}_p\!\cdot_q\, b) = \dot{\mathrm{tr}}(b \,{}_q\!\cdot_p\, a)$;
2. $\langle a|b\rangle$ is independent of $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$;
3. $\langle \cdot|\cdot\rangle$ is bilinear.

Proposition 3.3. Suppose that $p, q \in [1, \infty]$, $1/p + 1/q = 1$ and $a \in L^p(A, \varphi)$; then

$$\|a\|_p = \sup\{|\dot{\mathrm{tr}}(a \,{}_p\!\cdot_q\, b)| : b \in L^q(A, \varphi), \|b\|_q \leq 1\}.$$

Proposition 3.4. Let $p \in]1, \infty]$ and $1/p + 1/q = 1$.

1. Let $a \in L^p(A, \varphi)$; then φ_a defined by

$$\varphi_a(b) := \dot{\mathrm{tr}}(a\ {}_p\cdot_q\ b),\ b \in L^q(A, \varphi),$$

   is a bounded linear functional on $L^q(A, \varphi)$.
2. The mapping $a \mapsto \varphi_a$ is an isometric isomorphism of $L^p(A, \varphi)$ onto the dual Banach space of $L^q(A, \varphi)$.

Proposition 3.5. $(L^2(A, \varphi), \|\cdot\|_2)$ is a Hilbert space with the inner product

$$(a|b)_{L^2(A,\varphi)} := \dot{\mathrm{tr}}(b^*\ {}_2\cdot_2\ a)(= \dot{\mathrm{tr}}(a\ {}_2\cdot_2\ b^*))$$

for $a, b \in L^2(A, \varphi)$.

We turn now to interpolation. Let $L^p_\varphi(M)$ denote the Kosaki spaces defined by

$$L^p_\varphi(M) = h_\varphi^{1/2q} \cdot L^p(M) \cdot h_\varphi^{1/2q} \subset L^1(M),\ p, q \in [1, \infty],\ 1/p + 1/q = 1,$$

with the norm

$$\|h_\varphi^{1/2q} \cdot x \cdot h_\varphi^{1/2q}\|_p = \|x\|_p \text{ for } x \in L^p(M).$$

We know that

$$L^\infty(A, \varphi) \subset L^p(A, \varphi) \subset L^q(A, \varphi) \subset L^1(A, \varphi)$$

for $q \in [1, \infty]$, $q \le p$ and that $L^\infty(A, \varphi)$ and $L^1(A, \varphi)$ form a pair of compatible Banach spaces. Denote by $C_\theta(X_0, X_1)$ the Calderon's complex interpolation functor for the pair of compatible Banach spaces (X_0, X_1) (Berg and Löfström, 1976; Calderon, 1964; Kosaki, 1984). We refer now to the paper of Kosaki, 1984. Using his notation from sections 8, 9, we put $\phi_0 = \psi_0 = \omega$, $h_0 = k_0 = h_\varphi$ and $\eta = 1/2$ where ω, h_φ are our normal faithful state on M and the corresponding Radon-Nikodym derivative defined at the beginning of this section. We consider now the isometry

$$\Gamma_1 : L^1(A, \varphi) \to L^1(M),\ \Gamma_1|_{A/N_\varphi} : a \mapsto h_\varphi^{1/2} \cdot \Gamma_\infty(a) \cdot h_\varphi^{1/2}.$$

Then the restriction of Γ_1 to $L^p(A, \varphi)$ is an embedding of $L^p(A, \varphi)$ into the Haagerup $L^1(M)$ space such that $\Gamma_1(L^p(A, \varphi))$ is exactly the Kosaki complex interpolation space $C_{1/p}(M^{1/2}, M_*)$. In fact, it is clear that Γ_1 is an isometric isomorphisms from $L^1(A, \varphi)$ to $L^1_\varphi(M)$. We have

$$\begin{aligned}\Gamma_1(L^\infty(A, \varphi)) &= h_\varphi^{1/2} \cdot \Gamma_\infty(L^\infty(A, \varphi)) \cdot h_\varphi^{1/2} \\ &= i_1(L^\infty(M)) = L^\infty_\varphi(M).\end{aligned}$$

We easily check that Γ_1 restricted to $L^\infty(A, \varphi)$ takes the space isometrically onto $L^\infty_\varphi(M)$. For $p \in]1, \infty[$, let $q \in]1, \infty[$ be s.t. $1/p + 1/q = 1$; then, for $a \in A/N_\varphi$,

$$\gamma_1 (a) = h_\varphi^{1/2q} \cdot \gamma_p (a) \cdot h_\varphi^{1/2q}.$$

Thus

$$\Gamma_1 (L^p(A, \varphi)) = h_\varphi^{1/2q} \cdot L^p(M) \cdot h_\varphi^{1/2q} = L^p_\varphi(M).$$

It is routine to check that Γ_p is an isometric isomorphism from $L^p(A, \varphi)$ to $L^p_\varphi(M)$. We conclude the following.

Theorem 3.6. $C_{1/p}(L^1(A, \varphi), L^\infty(A, \varphi)) = L^p(A, \varphi)$, that is our L^p-spaces are interpolation spaces.

4. EXAMPLES

In view of possible applications (Majewski and Zegarliński, 1995, 1996), it is important to know how the L^p-spaces behave under inductive limits. We exhibit two situations in which they behave well.

Theorem 4.1. Let (A, α_t) be a C^*-dynamical system and $\{A_j\}_{j \in I}$ a generating nest of C^*-subalgebras of A, invariant under $\{\alpha_t\}$. Let φ be an α_t-KMS state on A. Then, for $p \in [1, \infty[$, $L^p(A, \varphi)$ is an inductive limit of $\{L^p(A_j, \varphi_j)\}_{j \in I}$ where $\varphi_j = \varphi|A_j$, $j \in I$ and, moreover, $L^p(A, \varphi) \cong L^p(\pi_\varphi(A)'')$.

Theorem 4.2. Let A be a UHF C^*-algebra with a generating nest $\{A_n\}$ and φ a product state on A with respect to the sequence $\{A_n\}$. Suppose that for each i, φ_i is faithful. Then for $p \in [1, \infty]$, $L^p(A, \varphi)$ is the inductive limit of $\{L^p (A_n, \varphi^{(n)})\}$; moreover, $L^p (A, \varphi) \cong L^p (\pi_\varphi (A)'')$.

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REFERENCES

Berg, J. and Löfström J. (1976) Interpolation spaces: An introduction, Springer Verlag, Berlin.

Calderon, A. P. (1964) Intermediate spaces and interpolation, the complex method, *Studia Mathematica*, **24**, 113–190.

Goldstein, S. and Lindsay, J. M. (1995) KMS symmetric Markov semigroups, *Mathematische Zeitschrift*, **219**, 591–608.

Goldstein, S. and Lindsay, J. M. (1999) Noncommutative interpolation and Markov semigroups, preprint.

Goldstein, S. and Phan V. T. (1998) L^p-spaces for UHF algebras, *International Journal of Theoretical Physics*, **37**, 593–598.

Kosaki, H. (1984) Application of the complex interpolation method to a von Neumann algebra: Non-commutative L^p-spaces, *Journal of Functional Analysis*, **56**, 29–78.

Majewski, A. W. and Zegarliński, B. (1995) Quantum stochastic dynamics I: Spin systems on a lattice, *Mathematical Physics Electronic Journal* **1** (2).

Majewski, A. W. and Zegarliński, B. (1996) On quantum stochastic dynamics and non-commutative L_p-spaces, *Letters in Mathematical Physics*, **36**, 337–349.

Phan V. T. (1999) L^p-spaces for C^*-algebras, Doctoral Thesis, University of Lódź.

Terp, M. (1982) Interpolation spaces between a von Neumann algebra and its predual, *Journal of Operator Theory* **8**, 327–360.